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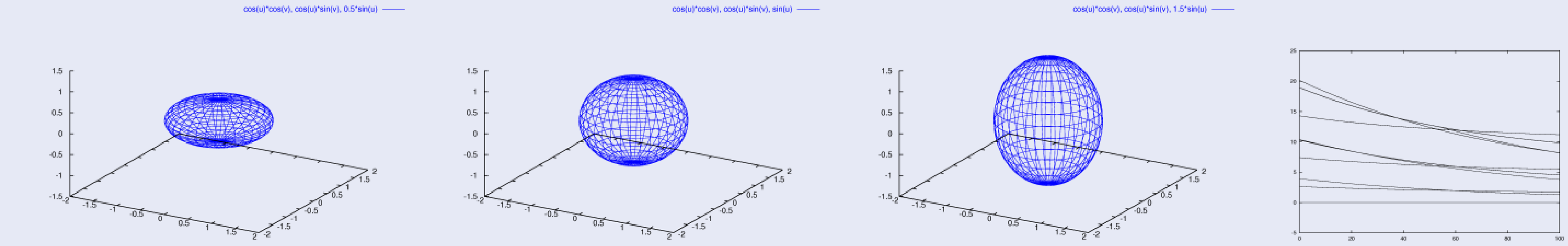
Eigenvalues of the Laplace-Beltrami operator for shape analysis

Can one hear the shape of a drum?



Deformation of a sphere and the resulting continuous shift of eigenvalues

The problem of shape characterization

- **Main goal**: Find a selected shape within a large collection of probably similar shapes
- **Secondary goal**: Find other similar shapes
- **Problem**: Classical features are often either insensitive to local or global changes, combining both (local and global features) is sometimes not straight forward
- Find a feature, that is both local **and** global

The spectrum of the Laplacian

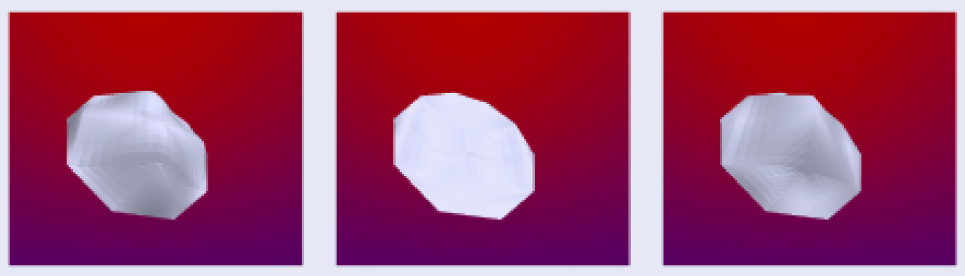
The **Laplacian** (also **Laplace-Beltrami operator**) on a manifold is defined by

$$\Delta f := \text{div grad } f$$

The **spectrum of the laplacian** of a manifold is given by the set of eigenvalues of the Laplacian on that manifold, i.e. by solutions λ of the equation

$$\Delta f + \lambda f = 0$$

Also known as **eigenfrequencies** because of their connection to oscillation problems of membranes $\rightsquigarrow$ **Can one hear the shape of a drum?** [L. Bers, cited in M. Kac, 1966]



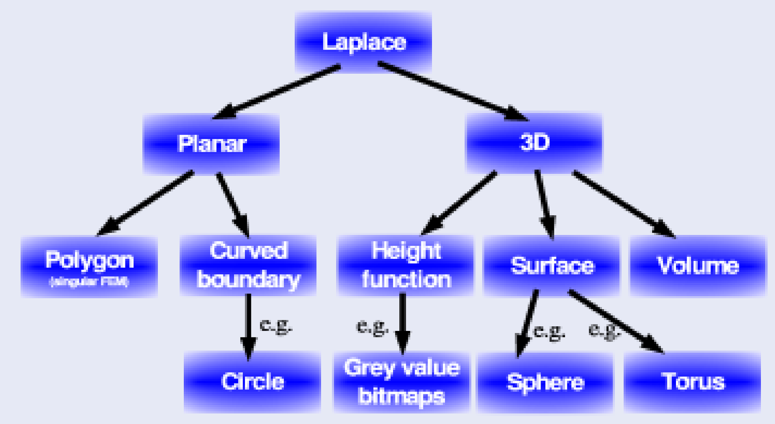
An oscillating membrane

Properties of the spectrum

- principally infinite, discrete: $0 \leq \lambda_1 < \lambda_2 < \ldots \uparrow +\infty$, each eigenspace is finite dimensional
- Isospectrality (i.e. two noncongruent shapes possessing identical spectra) is a rare phenomenon
- isometric invariant, thus insensitive to euclidean motions and reparametrisation
- depending continously on the shape
- sensitive to local **and** global changes of the shape; e.g. if a domain is scaled by s the eigenvalues are scaled by $\frac{1}{s^2}$
- determines geometrical and topological features, e.g. volume, boundary volume and Euler characteristic. For a planar polygonial domain with area A, boundary length U, corners of angle α_j and with $t \to 0$ this asymptotic expansion is known (McKean and Singer [1971], based on Weyl [1946]):

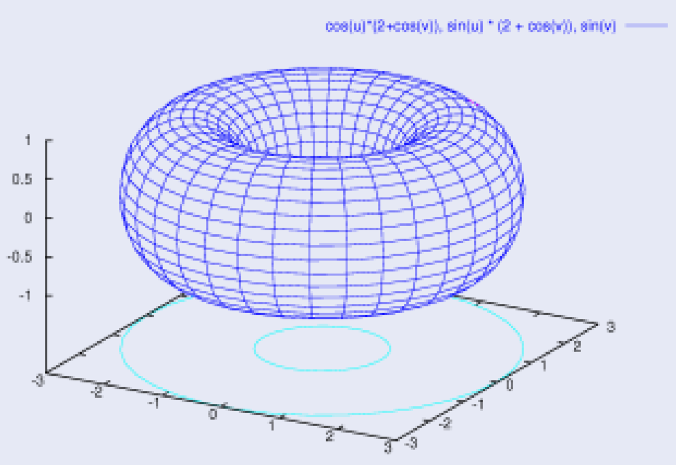
$$Z(t) = \sum_{n\geq 1} \exp(\lambda_n t) \sim \frac{A}{4\pi t} - \frac{U}{8\sqrt{\pi t}} + \sum_j \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j}$$

Scheme of implementations



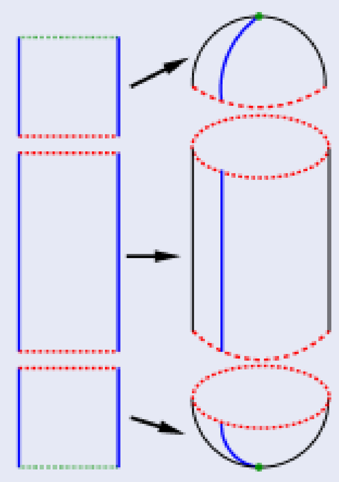
Computing the spectrum ... with classical FEM

- Works on arbitrary manifolds with linear, quadratic and cubic form functions
- Good results for "tame" surfaces, such as planar shapes or spheres, weaker results for highly curved surfaces (hard to mesh, hard to validate, as the analytical solutions are unknown)



Eigenvalues of the torus surface can be calculated with FEM

- Possible to weld object together to create an "atlas" (e.g. parametrisation of torus on rectangle is welded with itself on each side)
- Method to weld different objects also implemented. Resulting object exists of various parameter regions and the welding information

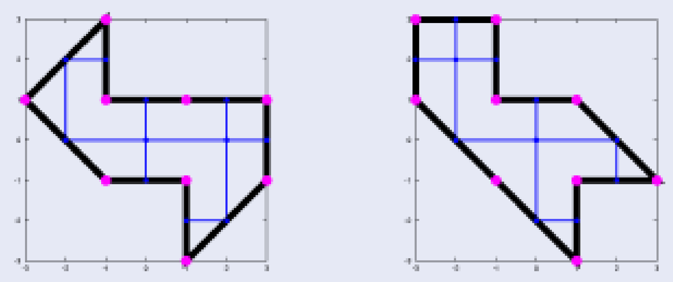


Atlas of cylinder with spherical caps

- Mesh in parameter space, calculation directly on surface
- Calculation possible on polyhedron and NURBS surfaces

Computing the spectrum ... with singular elements

- Singular elements initiated by Descloux, Tolley [1983] and Driscoll [1997]
- Best results for polygonal planar shapes
- Not yet applicable for other surfaces



Isospectral domains (GWW) with singular elements

- First 25 eigenvalues of isospectral GWW domains correspond with each other for at least 10 digits

Applications / Outlook

- indexing of large data sets of 2D- and 3D-shapes
- extraction of non-obvious geometrical and topological features
- implementation of fast solvers (Lanczos)
- error analysis
- adaption of singular elements to other surfaces

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